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University of Pennsylvania

CONCERNING THE INVARIANT THEORY OF INVOLUTIONS OF CONICS

A THESIS
PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

BY
WAYNE SENSENIG

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Concerning the Invariant Theory of Involutions of Conics.

By Wayne Sensenig.

§ 1. Introduction.

The complete system of invariants of a single conic

$$f=a_x^2=a_{200}x_1^2+2a_{110}x_1x_2+\dots.$$

consists of four members: $f=a_x^2$, $L=\alpha_u^2$, $D=a_\alpha^2$, and u_x. The complete system of invariants of two conics consists of twenty forms and this was derived first by Gordan and was first published in Clebsch's "Vorlesungen über Geometrie."* H. F. Baker derived the complete system for three, and incidentally two, conics.† Ciamberlini has published a determination of a complete system for three conics.‡

The object of this paper is to derive and reduce in terms of the system of two conics a_x^2, b_x^2 the complete simultaneous system of the involution $K=a_x^2+kb_x^2$ and the harmonic conic $F=(\alpha\beta x)^2$. The complete system of K and F is expressed in terms of the complete system of the two base conics $f=a_x^2$ and $g=b_x^2$, which determine the configuration.

In Bôchers "Introduction to Higher Algebra," p. 164, it is shown that if two conics f, g intersect in four distinct points there exists a (non-singular) collineation which will reduce f and g simultaneously into the normal forms given below. Note that the equalities hold only by virtue of the equations of the collineation:

$$a_x^2=ax_1^2+bx_2^2+cx_3^2, \quad b_x^2=px_1^2+qx_2^2+rx_3^2. \tag{1}$$

In Table I, I give the well-known symbolical forms of the system of f and g, and also the normal system which I have computed from these symbolical forms by particularizing the coefficients of f and g, respectively, according to the following scheme:

$$a_{110}=a_{101}=a_{011}=0, \quad a_{200}=a, \quad a_{020}=b, \quad a_{002}=c,$$
$$b_{110}=b_{101}=b_{011}=0, \quad b_{200}=p, \quad b_{020}=q, \quad b_{002}=r.$$

Throughout the paper we have used the algorism for ternary transvection given by O. E. Glenn in the Transactions of the American Mathematical Society,

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*Cf. Osgood, American Journal of Mathematics, Vol. XIV (1892), p. 262, "System of two Simultaneous Ternary Quadratic Forms."

† *Cambridge Philosophical Transactions*, Vol. XV (1889–93), p. 62.

‡ *Giornale di Matematiche*, Vol. XXIV (1885–86), p. 141.

Vol. XVII (1916).* This algorism is analytical but it will suffice here to describe the transvectant τ of index $\binom{i\,j}{k\,l}$ of the two forms

$$\phi = A a_{1x} a_{2x} \ldots . a_{rx} \alpha_{1u} \ldots . \alpha_{su}, \quad \psi = B b_{1x} \ldots . b_{px} \beta_{1u} \ldots . \beta_{qu},$$

where A and B are constants, as AB/μ times the sum of all terms gotten by forming in the product $\phi\psi$, in all possible (μ) ways, i convolutions of the type $a_{1\beta_1}$, j of type $(a_1 b_1 u)$, k of the type α_{1b_1}, and l of type $(\alpha_1 \beta_1 x)$. Then τ is abbreviated as

$$\tau = (\phi, \psi)^{ij}_{kl}.$$

In deriving the system of K and F we have evaluated each member as a transvectant and then reduced each form to a rational integral function of the members of the fundamental system of a_x^2 and b_x^2, expressing each one in the form of a polynomial in k. These reductions were performed by means of the following fundamental ternary identities, and their modifications:

$$\left.\begin{aligned} &(abc)\,d_x - (abd)\,c_x + (acd)\,b_x - (bcd)\,a_x = 0, \\ &(abu)\,(\alpha\beta x) = \begin{vmatrix} a_\alpha & b_\alpha & u_\alpha \\ a_\beta & b_\beta & u_\beta \\ a_x & b_x & u_x \end{vmatrix}. \end{aligned}\right\} \qquad (2)$$

The result for each case was then checked by direct verification, for the system of normal forms given in Table I. Table II contains the summary of the results of the reductions for all twenty concomitants in the system for K and F.

In Table III of the paper we give the fundamental system of the involution $a_x^2 + k b_x^2$ taken with a third general conic c_x^2. All concomitants in this system are expressed as rational integral functions of certain forms of the system for three conics as given by Baker; in fact in terms of the sixty-one forms of this system which are the simplest and therefore the most important in geometrical applications. This set of sixty-one concomitants is given below:

$f = a_x^2$, $g = b_x^2$, $\Phi_2 = (acu)^2$, $\Phi_1 = (bcu)^2$, $C_{12} = (acu)\,a_x c_x$, $C_{11} = (bcu)\,b_x c_x$,
$C_1 = a_x b_x (abu)$, $A_1 = a_\gamma^2$, $A_2 = b_\gamma^2$, $A_3 = c_\alpha^2$, $A_4 = c_\beta^2$, $C_{21} = a_\gamma a_x \gamma_u$,
$C_{22} = b_\gamma b_x \gamma_u$, $C_{23} = c_\alpha c_x \alpha_u$, $C_{24} = c_\beta c_x \beta_u$, $C_{42} = (\alpha\gamma x)\,\alpha_u \gamma_u$, $C_{41} = (\beta\gamma x)\,\beta_u \gamma_u$,
$F_2 = (\alpha\gamma x)^2$, $F_1 = (\beta\gamma x)^2$, $C_{51} = c_\alpha c_x \gamma_u (\alpha\gamma x)$, $C_{52} = c_\beta c_x \gamma_u (\beta\gamma x)$,
$C_{53} = a_\gamma a_x \alpha_u (\alpha\gamma x)$, $L'' = \gamma_u^2$, $C_{61} = a_\gamma c_x \gamma_u (acu)$, $C_{62} = b_\gamma c_x \gamma_u (bcu)$,
$C_{63} = a_x c_\alpha \alpha_u (acu)$, $C_{64} = b_x c_\beta \beta_u (bcu)$, $G_1 = a_\gamma c_\alpha a_x c_x (\alpha\gamma x)$, $G_2 = b_\gamma c_\beta b_x c_x (\beta\gamma x)$,
$\Gamma_2 = a_\gamma c_\alpha u_\alpha u_\gamma (acu)$, $\Gamma_1 = b_\gamma c_\beta u_\beta u_\gamma (bcu)$, $J = u_x$, $M = (abc)\,(abu)\,c_x$,
$N = (abc)^2$, $O_1 = a_\gamma b_x \gamma_u (abu)$, $O_2 = a_x b_\gamma \gamma_u (abu)$, $O_3 = b_x c_\alpha \alpha_u (bcu)$,
$O_4 = a_x c_\beta \beta_u (acu)$, $P = a_\gamma a_x b_\gamma b_x$, $Q_1 = a_\gamma b_x c_x \gamma_u (abc)$, $Q_2 = a_x b_\gamma c_x \gamma_u (abc)$,

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* Cf. Glenn, "Theory of Invariants" (Boston, Ginn & Co., 1915), p. 227.

$R_1=(a'bc)(acu)(abu)a'_x$, $R_2=(ab'c)(bcu)(abu)b'_x$, $S_1=b_\alpha c_\alpha(bcu)$,
$S_2=a_\beta c_\beta(acu)$, $S_3=a_\gamma b_\gamma(abu)$, $T_1=b_x b_\gamma \alpha_u(\alpha\gamma x)$, $T_2=b_x b_\alpha \gamma_u(\alpha\gamma x)$,
$T_3=a_\gamma a_x \beta_u(\beta\gamma x)$, $T_4=a_\beta a_x u_\gamma(\beta\gamma x)$, $T_5=b_\gamma b_x \beta_u(\beta\gamma x)$, $U_1=b_\alpha b_\gamma(\alpha\gamma x)$,
$U_2=a_\beta a_\gamma(\beta\gamma x)$, $V=a_x b_x c_x(abc)$, $W=a_\gamma b_\gamma c_x(abc)$, $X_1=b_\gamma c_\alpha b_x c_x(\alpha\gamma x)$,
$X_2=a_\gamma c_\beta a_x c_x(\beta\gamma x)$, $Y=(acu)(abu)(bcu)$, $Z_1=b_\gamma c_\alpha u_\alpha u_\gamma(bcu)$, $Z_2=a_\gamma c_\beta u_\beta u_\gamma(acu)$;
where those preceding J are concomitants of two conics and those following J, of three conics.

Professor Glenn suggested the problem of this paper to me and gave me much valuable advice during the investigation.

TABLE I.

Normal System of Two Conics.

1. $f=a_x^2=a_x^{\prime 2}=a_x^{\prime\prime 2}=\text{etc.}=ax_1^2+bx_2^2+cx_3^2.$

2. $g=b_x^2=b_x^{\prime 2}=b_x^{\prime\prime 2}=\text{etc.}=px_1^2+qx_2^2+rx_3^2.$

3. $D=a_\alpha^2=6abc.$

4. $D'=b_\beta^2=6pqr.$

5. $L=\alpha_u^2=2(bcu_1^2+acu_2^2+abu_3^2).$

6. $L'=\beta_u^2=2(qru_1^2+pru_2^2+pqu_3^2).$

7. $\Phi=(a_x^2, b_x^2)^{02}_{00}=(abu)^2=(br+cq)u_1^2+(ar+cp)u_2^2+(aq+bp)u_3^2.$

8. $C_1=(a_x^2, b_x^2)^{01}_{00}=(abu)a_x b_x$
$=(aq-bp)x_1x_2u_3+(cp-ar)x_1x_3u_2+(br-cq)x_2x_3u_1.$

9. $A_{122}=(a_x^2, \beta_u^2)^{20}_{00}=a_\beta^2=2(aqr+bpr+cpq).$

10. $C_2=(a_x^2, \beta_u^2)^{10}_{00}=a_\beta a_x \beta_u=2(aqrx_1u_1+bprx_2u_2+cpqx_3u_3).$

11. $C_3=(\alpha_u^2, b_x^2)^{00}_{10}=\alpha_b \alpha_u b_x=2(bcpx_1u_1+acqx_2u_2+abrx_3u_8).$

12. $A_{112}=(\alpha_u^2, b_x^2)^{00}_{20}=\alpha_b^2=2(bcp+acq+abr).$

13. $C_4=(\alpha_u^2, \beta_u^2)^{00}_{01}=(\alpha\beta x)\alpha_u\beta_u$
$=4[cr(bp-aq)u_1u_2x_3+bq(ar-cp)u_1u_3x_2+ap(cq-br)u_2u_3x_1].$

14. $F=(\alpha_u^2, \beta_u^2)^{00}_{02}=(\alpha\beta x)^2$
$=4[ap(br+cq)x_1^2+bq(ar+cp)x_2^2+cr(bp+aq)x_3^2].$

15. $C_5=(\alpha_u^2, b_x^2\beta_u^2)^{00}_{11}=\alpha_b(\alpha\beta x)b_x\beta_u$
$=4[aqr(br-cq)x_2x_3u_1+bpr(cp-ar)x_1x_3u_2+cpq(aq-bp)x_1x_2u_8].$

16. $C_6=(a_x^2, b_x^2\beta_u^2)^{11}_{00}=a_\beta(abu)b_x\beta_u$
$=2[p^2(cq-br)u_2u_3x_1+q^2(ar-cp)u_1u_3x_2+r^2(bp-aq)u_1u_2x_8].$

17. $C_7=(a_x^2\alpha_u^2, b_x^2)^{01}_{10}=\alpha_b(abu)a_x\alpha_u$
$=2[a^2(cq-br)u_2u_3x_1+b^2(ar-cp)u_1u_3x_2+c^2(bp-aq)u_1u_2x_8].$

18. $C_8=(a_x^2\alpha_u^2, \beta_u^2)^{10}_{01}=a_\beta(\alpha\beta x)a_x\alpha_u$
$=4[abr(aq-bp)x_1x_2u_3+acq(cp-ar)x_1x_3u_2+bcp(br-cq)x_2x_3u_1].$

19. $G=(fL, gL')^{10}_{11}=a_\beta\alpha_b(\alpha\beta x)a_x b_x$
$=4[a^2qr(br-cq)+b^2pr(cp-ar)+c^2pq(aq-bp)]x_1x_2x_3.$

20. $\Gamma=(fL, gL')^{11}_{10}=a_\beta\alpha_b(abu)\alpha_u\beta_u$
$=4[p^2bc(cq-br)+q^2ac(ar-cp)+r^2ab(bp-aq)]u_1u_2u_8.$

§ 2. REDUCTION OF THE SYSTEM OF K AND F.

The complete system of K and F is obtained by forming and reducing the transvectants of K and F which correspond respectively to the transvectants of f, g required to produce the system of f, g (cf. Table I). Thus a member of the system of f, g is $\Phi=(f, g)^{02}_{00}$. Hence the corresponding form of the required system is

$$\Phi'=(K, F)^{02}_{00}.$$

I have not included in this paper all of the reductions which were necessary to produce Table II; but only typical ones. The most complicated cases of all were C'_5, G', and Γ'. We add that the reductions for 3, 4, 5, 6 in Table II had been given previously.*

The Form Φ'.

$$\begin{aligned}\Phi'=(K, F)^{02}_{00}&=(a^2_x+kb^2_x,\ (\alpha\beta)^2_x)^{02}_{00}=(a^2_x,\ (\alpha\beta)^2_x)^{02}_{00}+k(b^2_x,\ (\alpha\beta)^2_x)^{02}_{00}\\ &=(a\overgroup{\alpha\beta}u)^2+k(b\overgroup{\alpha\beta}u)^2=(a_\alpha u_\beta-a_\beta u_\alpha)^2+k(b_\alpha u_\beta-b_\beta u_\alpha)^2\\ &=a^2_\alpha u^2_\beta+a^2_\beta u^2_\alpha-2a_\alpha a_\beta u_\alpha u_\beta+k(b^2_\alpha u^2_\beta+b^2_\beta u^2_\alpha-2b_\alpha b_\beta u_\alpha u_\beta)\\ &=DL'+A_{122}L-2(aa'a'')(a'a''u)a_\beta u_\beta\\ &\qquad+k[D'L+A_{112}L'-2(bb'b'')(b'b''u)b_\alpha u_\alpha]\\ &=DL'+A_{122}L-\tfrac{2}{3}DL'+k[D'L+A_{112}L'-\tfrac{2}{3}D'L]\\ &=\tfrac{1}{3}DL'+A_{122}L+k[\tfrac{1}{3}D'L+A_{112}L'].\end{aligned}$$

The Form C'_3.

$$\begin{aligned}C'_3&=(\alpha^2_u+2k(abu)^2+k^2\beta^2_u,\ (\gamma\delta)^2_x)^{00}_{10},\ \text{where } \alpha=\gamma \text{ and } \beta=\delta,\\ &=(\alpha^2_u,\ (\gamma\delta)^2_x)^{00}_{10}+2k[(ab)^2_u,\ (\gamma\delta)^2_x]^{00}_{10}+k^2[\beta^2_u,\ (\gamma\delta)^2_x]^{00}_{10}\\ &=(\alpha\gamma\delta)\alpha_u(\gamma\delta x)+2k(\overgroup{ab}\gamma\delta)(abu)(\gamma\delta x)+k^2(\beta\gamma\delta)\beta_u(\gamma\delta x).\end{aligned}$$

1. $(\alpha\gamma\delta)\alpha_u(\gamma\delta x)\equiv\tfrac{1}{2}(\alpha\gamma\delta)\{(\gamma\delta x)\alpha_u-(\alpha\delta x)\gamma_u\}=\tfrac{1}{2}(\alpha\gamma\delta)[(\alpha\gamma\delta)x_u-(\alpha\gamma x)\delta_u]$

$$=\tfrac{1}{2}(\alpha\gamma\delta)^2x_u-\tfrac{1}{2}(\alpha\gamma\delta)(\alpha\gamma x)\delta_u.$$

$$\begin{aligned}(\alpha\gamma\delta)(\alpha\gamma x)\delta_u&=(\alpha\overgroup{a'''a^{\mathrm{IV}}}\delta)(\alpha\overgroup{a'''a^{\mathrm{IV}}}x)\delta_u=(a'''_\alpha a^{\mathrm{IV}}_\delta-a^{\mathrm{IV}}_\alpha a'''_\delta)(a'''_\alpha a^{\mathrm{IV}}_x-a^{\mathrm{IV}}_\alpha a'''_x)\delta_u\\ &=[(a'a''a''')(b'b''a^{\mathrm{IV}})-(a'a''a^{\mathrm{IV}})(b'b''a''')][(a'a''a''')a^{\mathrm{IV}}_x-(a'a''a^{\mathrm{IV}})a'''_x]\delta_u\\ &=[(a'a''a''')^2a^{\mathrm{IV}}_\beta a^{\mathrm{IV}}_x-(a'a''a''')(a'a''a^{\mathrm{IV}})a^{\mathrm{IV}}_\beta a'''_x\\ &\qquad+(a'a''a^{\mathrm{IV}})^2a'''_\beta a'''_x-(a'a''a''')(a'a''a^{\mathrm{IV}})a'''_\beta a^{\mathrm{IV}}_x]\delta_u\\ &=D\cdot a^{\mathrm{IV}}_\beta a^{\mathrm{IV}}_x\beta_u-\tfrac{1}{3}D\cdot a^{\mathrm{IV}}_\beta a^{\mathrm{IV}}_x\beta_u+D\cdot a'''_\beta a'''_x\beta_u-\tfrac{1}{3}D\cdot a^{\mathrm{IV}}_\beta a^{\mathrm{IV}}_x\beta_u=\tfrac{4}{3}DC_2.\end{aligned}$$

$\therefore\ (\alpha\gamma\delta)(\gamma\delta x)\alpha_u=\tfrac{2}{3}DA_{122}u_x-\tfrac{2}{3}DC_2$; since $(\alpha\gamma\delta)^2=\tfrac{4}{3}DA_{122}$, $(\beta\gamma\delta)^2=\tfrac{4}{3}D'A_{112}$.

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* "Algebra of Invariants," by Grace and Young, p. 293, *Cambridge University Press*, 1903.

2. $(\overset{\frown}{ab}\gamma\delta)(abu)(\gamma\delta x)=(a_\gamma b_\delta-a_\delta b_\gamma)(abu)(\alpha\beta x)=a_\alpha b_\beta(abu)(\alpha\beta x)$

$\quad -a_\beta b_\alpha(abu)(\alpha\beta x)=a_\alpha b_\beta(abu)(\alpha\beta x)-k_3$

$\quad =\tfrac{2}{9}DD'u_x-k_3$, since $a_\alpha b_\beta(abu)(\overset{\frown}{\alpha b'b''}x)=a_\alpha b_\beta(abu)[b''_\alpha b'_x-b'_\alpha b''_x]$

$\quad =a_\alpha b''_\alpha b_\beta b'_x(abu)-a_\alpha b'_\alpha b_\beta b''_x(abu)=(aa'a'')(a'a''b'')a_{(bu)}b_\beta b'_x$

$\quad -(aa'a'')(a'a''b')a_{(bu)}b_\beta b''_x=\tfrac{1}{3}D(b''bu)(bb'b'')b'_x$

$\quad -\tfrac{1}{3}D(b'bu)(bb'b'')b''_x=\tfrac{1}{3}D(bb''u)(bb''b')b'_x+\tfrac{1}{3}D(bb'u)(bb'b'')b''_x$

$\quad =\tfrac{1}{9}DD'u_x+\tfrac{1}{9}DD'u_x=\tfrac{2}{9}DD'u_x.$

3. $(\beta\gamma\delta)(\gamma\delta x)\beta_u=\tfrac{1}{2}(\beta\gamma\delta)[(\gamma\delta x)\beta_u+(\beta\gamma x)\delta_u]=\tfrac{1}{2}(\beta\gamma\delta)[(\gamma\delta x)\beta_u+(\beta\gamma x)\delta_u]$

$\quad \equiv\tfrac{1}{2}(\beta\gamma\delta)[(\beta\gamma\delta)x_u+(\beta\delta x)\gamma_u]=\tfrac{1}{2}(\beta\gamma\delta)^2x_u+\tfrac{1}{2}(\beta\gamma\delta)(\beta\delta x)\gamma_u.$

$\tfrac{1}{2}(\beta\gamma\delta)(\beta\delta x)\gamma_u=\tfrac{1}{2}(\overset{\frown}{a'a''}\beta'\beta)(\overset{\frown}{b'b''}\beta'x)\alpha_u=\tfrac{1}{2}(a'_{\beta'}a''_\beta-a'_\beta a''_{\beta'})(b'_{\beta'}b''_x-b'_xb''_{\beta'})\alpha_u$

$\quad =\tfrac{1}{2}[a'_{\beta'}a''_\beta b'_{\beta'}b''_x\alpha_u-a'_{\beta'}a''_\beta b'_xb''_{\beta'}\alpha_u-a'_\beta a''_{\beta'}b'_{\beta'}b''_x\alpha_u+a'_\beta a''_{\beta'}b'_xb''_{\beta'}\alpha_u]$

$\quad =2a'_{\beta'}a''_\beta b'_{\beta'}b''_x\alpha_u=2(b'b'''b^{\text{IV}})(b'''b^{\text{IV}}a')b'_{(b''a'')}b''_x\alpha_u=\tfrac{2}{3}D'(a'b''a'')b''_x\alpha_u$

$\quad =-\tfrac{2}{3}D'b''_\alpha b''_x\alpha_u=-\tfrac{2}{3}D'C_3.$

$\therefore(\beta\gamma\delta)(\gamma\delta x)\beta_u=\tfrac{2}{3}D'A_{112}u_x-\tfrac{2}{3}D'C_3.\quad k_3=a_\beta b_\alpha(abu)(\alpha\beta x)=\tfrac{1}{9}DD'u_x$

$\quad +A_{112}C_2-A_{112}A_{122}u_x+A_{122}C_3.$

$\therefore C'_3=\tfrac{2}{3}DA_{122}u_x-\tfrac{2}{3}DC_2+2k\{\tfrac{1}{9}DD'u_x-A_{112}C_2+A_{112}A_{122}u_x-A_{122}C_3\}$

$\quad +k^2(\tfrac{2}{3}D'A_{112}u_x-\tfrac{2}{3}D'C_3).$

The Form C'_7.

$C'_7=\{(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2),(\alpha\beta)_x^2\}^{01}_{10}=\{a_x^2\alpha_u^2,(\alpha'\beta)_x^2\}^{01}_{10}$

$\quad +2k\{a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\}^{01}_{10}+k^2\{a_x^2\beta_u^2,(\alpha\beta')_x^2\}^{01}_{10}+k\{b_x^2\alpha_u^2,(\alpha'\beta)_x^2\}^{01}_{10}$

$\quad +2k^2\{b_x^2(ab''')_u^2,(\alpha\beta)_x^2\}^{01}_{10}+k^3\{b_x^2\beta_u^2,(\alpha\beta')_x^2\}^{01}_{10}.$

1. $\{a_x^2\alpha_u^2,(\alpha'\beta)_x^2\}^{01}_{10}=(\alpha\alpha'\beta)(\overset{\frown}{a\alpha'\beta}u)a_x\alpha_u=(\alpha\alpha'\beta)\{a_\beta\alpha'_u-a_{\alpha'}\beta_u\}a_x\alpha_u$

$\quad =-(\alpha\alpha'\beta)a_{\alpha'}\beta_ua_x\alpha_u=-(aa'''a^{\text{IV}})(a'''a^{\text{IV}}\overset{\frown}{\beta\alpha})a_x\alpha_u\beta_u$

$\quad =-\tfrac{1}{3}D(\beta\alpha x)\alpha_u\beta_u=\tfrac{1}{3}DC_4.$

2. $\{a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\}^{01}_{10}=(\overset{\frown}{a'''b}\alpha\beta)(a\overset{\frown}{\alpha\beta}u)a_x(a'''bu)$

$\quad =(a'''_\alpha b_\beta-a'''_\beta b_\alpha)(a_\beta\alpha_u-a_\alpha\beta_u)a_x(a'''bu)=-a'''_\beta b_\alpha a_\beta\alpha_ua_x(a'''bu)$

$\qquad +a'''_\beta b_\alpha a_\alpha\beta_ua_x(a'''bu)\quad$ (Terms containing $a'''_\alpha b_\beta$ vanish.)

$\quad =(-A_{122}C_7+\tfrac{1}{2}A_{112}C_4-\tfrac{1}{3}DC_6)+\tfrac{1}{3}DC_6=\tfrac{1}{2}A_{112}C_4-A_{122}C_7.$

For, $a'''_\beta a_\beta a_xb_\alpha\alpha_u(a'''bu)-a'''^2_\beta\cdot a_xb_\alpha\alpha_u(abu)$

$\quad =a'''_\beta a_xb_\alpha\alpha_u\{(a'''bu)a_\beta-(abu)a'''_\beta\}\equiv a'''_\beta a_xb_\alpha\alpha_u[(aa'''b)u_\beta-(aa'''u)b_\beta]$

$\quad =\tfrac{1}{2}b_\alpha\alpha_uu_\beta(aa'''b)[a_xa'''_\beta-a_\beta a'''_x]-(bb'b'')(b'b''a''')b_\alpha a_x\alpha_u(aa'''u)$

$\quad =\tfrac{1}{2}b_\alpha\alpha_uu_\beta\bar{\alpha}_b(\alpha x\beta)-\tfrac{1}{3}D'(a'''a'a'')(a'a''u)a'''_{(ua)}a_x=\tfrac{1}{2}\alpha_b\alpha_u\alpha_b\bar{\alpha}_{(x\beta)}u_\beta.$

But, $\alpha_b\alpha_u\bar{\alpha}_b\bar{\alpha}_{(x\beta)}u_\beta-\alpha_b^2\cdot\alpha_u u_\beta(\alpha x\beta)=\bar{\alpha}_b\alpha_u u_\beta[(\bar{\alpha}_x\beta)\alpha_b-(\alpha x\beta)\bar{\alpha}_b]$

$\equiv\bar{\alpha}_b\alpha_u u_\beta[(\alpha\bar{\alpha}x)b_\beta-(\alpha\bar{\alpha}\beta)b_x]\equiv(bb'b'')(b'b''u)\bar{\alpha}_b\alpha_u(\alpha\bar{\alpha}x)-\bar{\alpha}_b\alpha_u u_\beta(\alpha\bar{\alpha}\beta)b_x$

$=-\bar{\alpha}_b\alpha_u u_\beta b_x[a''_\beta a'_{\bar{\alpha}}-a'_\beta a''_{\bar{\alpha}}]=\bar{\alpha}_{a''}\bar{\alpha}_b a''_{(ua')}a'_\beta b_x u_\beta-\bar{\alpha}_{a'}\bar{\alpha}_b a'_{(a''u)}b_x u_\beta a''_\beta$

$=\tfrac{1}{3}D(bua')a'_\beta b_x u_\beta-\tfrac{1}{3}D(ba''u)b_x a''_\beta u_\beta=\tfrac{2}{3}D(abu)a_\beta b_x u_\beta=\tfrac{2}{3}DC_6.$

$\therefore a'''_\beta a_\beta a_x b_a\alpha_u(a'''bu)=A_{122}C_7-\tfrac{1}{2}A_{112}C_4+\tfrac{1}{3}DC_6.$

3. $\{a_x^2\beta_u^2,(\alpha\beta')_x^2\}_{10}^{01}=(\beta\alpha\beta')(\overset{\frown}{a\alpha\beta'}u)a_x\beta_u=(\beta\alpha\beta')\{a_{\beta'}\alpha_u-a_a\beta'_u\}a_x\beta_u$

$=a_{\beta'}\alpha_u a_x\beta_u\{b'_a b''_{\beta'}-b'_{\beta'}b''_a\}-\tfrac{1}{3}D(\beta'\beta x)\beta'_u\beta_u=(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a)b''_{(ub')}\alpha_u a_x b'_a$

$-(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a)b'_{(b''u)}\alpha_u a_x b''_a-0=-\tfrac{2}{3}D'C_7.$

4. $\{b_x^2\alpha_u^2,(\alpha'\beta)_x^2\}_{10}^{01}=(\alpha\alpha'\beta)(\overset{\frown}{b\alpha'\beta}u)b_x\alpha_u=(\alpha\alpha'\beta)\{b_\beta\alpha'_u-\alpha'_b\beta_u\}b_x\alpha_u$

$=b_\beta\alpha'_u b_x\alpha_u(\alpha\alpha'\beta)-b_{\alpha'}\beta_u b_x\alpha_u(\alpha\alpha'\beta)=0-b_{\alpha'}\beta_u b_x\alpha_u\{a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}\}=\tfrac{2}{3}DC_6.$

5. $\{b_x^2(ab''')_u^2,(\alpha\beta)_x^2\}_{10}^{01}=(\overset{\frown}{ab'''}\alpha\beta)(b\overset{\frown}{\alpha\beta}u)b_x(ab'''u)$

$=(a_a b'''_\beta-a_\beta b'''_a)(b_\beta\alpha_u-b_a\beta_u)b_x(ab'''u)=0-0-a_\beta b'''_a b_\beta\alpha_u b_x(ab'''u)$

$+a_\beta b'''_a b_a\beta_u b_x(ab'''u)^*=-\tfrac{1}{3}D'C_7+[A_{112}C_6-\tfrac{1}{2}A_{122}C_4+\tfrac{1}{3}D'C_7\}$

$=A_{112}C_6-\tfrac{1}{2}A_{122}C_4$; * similar to 2.

6. $\{b_x^2\beta_u^2,(\alpha\beta')_x^2\}_{10}^{01}=(\beta\alpha\beta')(\overset{\frown}{b\alpha\beta'}u)b_x\beta_u=(b'_a b''_{\beta'}-b'_{\beta'}b''_a)(b_{\beta'}\alpha_u-b_a\beta'_u)b_x\beta_u$

$=b'_a b''_{\beta'}b_{\beta'}\alpha_u b_x\beta_u{}^{**}-b'_a b''_{\beta'}b_a\beta'_u b_x\beta_u-b'_{\beta'}b''_a b_{\beta'}\alpha_u b_x\beta_u+b'_{\beta'}b''_a b_a\beta'_u b_x\beta_u.$

${}^{**}=(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}b)b''_{(ub')}b'_a b_x\alpha_u=\tfrac{1}{3}D'(bub')b'_a b_x\alpha_u$

$=\tfrac{1}{6}D'(bub')\alpha_u[b'_a b_x-b'_x b_a]=\tfrac{1}{6}D'(b'bu)\alpha_u(\overset{\frown}{b'b}\alpha x)$

$=\tfrac{1}{6}D'\beta'''_u\alpha_u(\beta'''\alpha x)=-\tfrac{1}{6}D'C_4.$

Also, $b'_{\beta'}b''_a b_{\beta'}\alpha_u b_x\beta_u={}^{**}$ by $b'\sim b''$, and $-b'_a b''_{\beta'}b_a\beta'_u b_x\beta_u=b'_{\beta'}b''_a b_a\beta'_u b_x\beta_u=0.$

Hence $\{b_x^2\beta_u^2,(\alpha\beta')_x^2\}_{10}^{01}=-\tfrac{1}{3}D'C_4.$

$\therefore C'_7=\tfrac{1}{3}DC_4+k\{A_{112}C_4-2A_{122}C_7+\tfrac{2}{3}DC_6\}+k^2\{-A_{122}C_4+A_{112}C_6-\tfrac{2}{3}D'C_7\}$

$+k^3(-\tfrac{1}{3}D'C_4).$

The Form G'.

$G'=[(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2),(\alpha\beta)_x^2(3a_a^2a_\beta^2\beta_u^2+3b_\beta^2b_a^2\alpha_u^2-2a_a^2b_\beta^2(ab)_u^2]_{11}^{10}$

$=3a_a^2a_\beta^2[(a_x^2\alpha_u^2,(\alpha'\beta)_x^2\beta_u'^2)_{11}^{10}+2k(a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\beta_u'^2)_{11}^{10}+k^2(a_x^2\beta_u^2,(\alpha\beta')_x^2\beta_u''^2)_{11}^{10}$

$+k(b_x^2\alpha_u^2,(\alpha'\beta)_x^2\beta_u'^2)_{11}^{10}+2k^2(b_x^2(ab''')_u^2,(\alpha\beta)_x^2\beta_u'^2)_{11}^{10}+k^3(b_x^2\beta_u^2,(\alpha\beta')_x^2\beta_u''^2)_{11}^{10}]$

$+3b_\beta^2b_a^2[(a_x^2\alpha_u^2,(\alpha'\beta)_x^2\alpha_u''^2)_{11}^{10}+2k(a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\alpha_u'^2)_{11}^{10}$

$+k^2(a_x^2\beta_u^2,(\alpha\beta')_x^2\alpha_u'^2)_{11}^{10}+k(b_x^2\alpha_u^2,(\alpha'\beta)_x^2\alpha_u''^2)_{11}^{10}+2k^2(b_x^2(ab''')_u^2,(\alpha\beta)_x^2\alpha_u'^2)_{11}^{10}$

$+k^3(b_x^2\beta_u^2,(\alpha\beta')_x^2\alpha_u'^2)_{11}^{10}]-2a_a^2b_\beta^2[(a_x^2\alpha_u^2,(\alpha'\beta)_x^2(a^{\mathrm{V}}b)_u^2)_{11}^{10}$

$+2k(a_x^2(a'''b)_u^2,(\alpha\beta)_x^2(a^{\mathrm{IV}}b''')_u^2)_{11}^{10}+k^2(a_x^2\beta_u^2,(\alpha\beta')_x^2(a'''b)_u^2)_{11}^{10}$

$+k(b_x^2\alpha_u^2,(\alpha'\beta)_x^2(ab''')_u^2)_{11}^{10}+2k^2(b_x^2(ab''')_u^2,(\alpha\beta)_x^2(a'''b^{\mathrm{IV}})_u^2)_{11}^{10}$

$+k^3(b_x^2\beta_u^2,(\alpha\beta')_x^2(ab^{\mathrm{V}})_u^2)_{11}^{10}].$

$$=3a_\alpha^2a_\beta^2[a_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)a_x(\alpha'\beta x)+2ka_{\beta'}(\overparen{a'''b}\alpha\beta)(\overparen{a'''b}\beta'x)a_x(\alpha\beta x)$$
$$+k^2a_{\beta''}(\beta\alpha\beta')(\beta\beta''x)a_x(\alpha\beta'x)+kb_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)b_x(\alpha'\beta x)$$
$$+2k^2b_{\beta'}(\overparen{ab'''}\alpha\beta)(\overparen{ab'''}\beta'x)b_x(\alpha\beta x)+k^3b_{\beta''}(\beta\alpha\beta')(\beta\beta''x)b_x(\alpha\beta'x)]$$
$$+3b_\beta^2b_\alpha^2[a_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)a_x(\alpha'\beta x)+2ka_{\alpha'}(\overparen{a'''b}\alpha\beta)(\overparen{a'''b}\alpha'x)a_x(\alpha\beta x)$$
$$+k^2a_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)a_x(\alpha\beta'x)+kb_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)b_x(\alpha'\beta x)$$
$$+2k^2b_{\alpha'}(\overparen{ab'''}\alpha\beta)(\overparen{ab'''}\alpha'x)b_x(\alpha\beta x)+k^3b_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)b_x(\alpha\beta'x)]$$
$$-2a_\alpha^2b_\beta^2[(aa^{\mathrm{V}}b)(\alpha\alpha'\beta)(\alpha\overparen{a^{\mathrm{V}}b}x)a_x(\alpha'\beta x)$$
$$+2k(aa^{\mathrm{IV}}b''')(\overparen{a'''b}\alpha\beta)(\overparen{a'''b}\overparen{a^{\mathrm{IV}}b'''}x)a_x(\alpha\beta x)$$
$$+k^2(aa'''b)(\beta\alpha\beta')(\beta\overparen{a'''b}x)a_x(\alpha\beta'x)+k(bab''')(\alpha\alpha'\beta)(\alpha\overparen{ab'''}x)b_x(\alpha'\beta x)$$
$$+2k^2(ba'''b^{\mathrm{IV}})(\overparen{ab'''}\alpha\beta)(\overparen{ab'''}\overparen{a'''b^{\mathrm{IV}}}x)b_x(\alpha\beta x)$$
$$+k^3(bab^{\mathrm{V}})(\beta\alpha\beta')(\beta\overparen{ab^{\mathrm{V}}}x)b_x(\alpha\beta'x)].$$
$$=3D\,A_{122}[0+2k(-A_{122}G)+k^2(-\tfrac{2}{3}D'G)+k(0)+2k^2(0)+k^3(0)]$$
$$+3D'A_{112}[0+2k(0)+k^2(0)+k(\tfrac{2}{3}DG)+2k^2(A_{112}G)+k^3(0)]$$
$$-2DD'[-\tfrac{1}{3}DG+2k(-\tfrac{1}{2}A_{112}G)+k^2(0)+k(0)$$
$$+2k^2(\tfrac{1}{2}A_{122}G)+k^3(\tfrac{1}{3}D'G)]. \qquad (3)$$

Collecting terms,

$$G'=\{\tfrac{2}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)+k^2(6D'A_{112}^2-4DD'A_{122})-\tfrac{2}{3}DD'^2\cdot k^3\}G.$$

The calculations are shown below:

1. $a_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)a_x(\alpha'\beta x)=\tfrac{1}{2}a_{\beta'}a_x(\alpha\alpha'\beta)[(\alpha\beta'x)\alpha'_{(\beta x)}-(\alpha'\beta'x)\alpha_{(\beta x)}]$
   $\equiv\tfrac{1}{2}a_{\beta'}a_x(\alpha\alpha'\beta)(\alpha\alpha'x)(\beta'\beta x)=\tfrac{1}{4}a_x(\alpha\alpha'x)(\beta\beta'x)[(\alpha\alpha'\beta)\beta'_\alpha-(\alpha\alpha'\beta')\beta_\alpha]$
   $\equiv\tfrac{1}{4}a_x(\alpha\alpha'x)(\beta\beta'x)[(\alpha'\beta\beta')a_\alpha-(\alpha\beta\beta')a_{\alpha'}]=0.$

2. $b_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)b_x(\alpha'\beta x)=(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\overparen{x\alpha})b_x(\alpha\alpha'\beta)(\alpha'\beta x)$
   $=\tfrac{1}{3}D'(x\alpha x)$, etc. $=0$.

3. $a_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)a_x(\alpha'\beta x)=(aa^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}\overparen{x\alpha})a_x$, etc. $=0$.

4. $b_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)b_x(\alpha'\beta x)=b_{\alpha''}(\alpha\alpha'\beta)[a'_{\alpha''}a''_x-a'_xa''_{\alpha''}]b_x(\alpha'\beta x)$
   $=b_{\alpha''}(\alpha\alpha'\beta)a'_{\alpha''}a''_xb_x(\alpha'\beta x)-b_{\alpha''}(\alpha\alpha'\beta)a'_xa''_{\alpha''}b_x(\alpha'\beta x)$
   $=(a'a^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}b)[a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}]a''_xb_x(\alpha'\beta x)$
   $-(a''a^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}b)[a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}]a'_xb_x(\alpha'\beta x)=\tfrac{1}{3}Db_{\alpha'}a''_\beta a''_xb_x(\alpha'\beta x)$
   $-\tfrac{1}{3}Db_\beta a''_{\alpha'}a''_xb_x(\alpha'\beta x)-\tfrac{1}{3}Db_\beta a'_{\alpha'}a'_xb_x(\alpha'\beta x)+\tfrac{1}{3}Db_{\alpha'}a'_\beta a'_xb_x(\alpha'\beta x)$
   $=\tfrac{2}{3}Db_{\alpha'}a''_\beta a''_xb_x(\alpha'\beta x)-\tfrac{2}{3}D(bb'b'')(b'b''\overparen{x\alpha'})b_xa''_xa''_{\alpha'}=\tfrac{2}{3}DG-0.$

5. $(aa^{\mathrm{V}}b)(\alpha\alpha'\beta)(\alpha\overparen{a^{\mathrm{V}}b}x)a_x(\alpha'\beta x)$

$$=\{a'_{\alpha'}a''_{\beta}-a'_{\beta}a''_{\alpha'}\}(\alpha'\beta x)\{a^{\mathrm{V}}_x b_\alpha-a^{\mathrm{V}}_\alpha b_x\}a_x(aa^{\mathrm{V}}b)=a'_{\alpha'}a''_{\beta}a^{\mathrm{V}}_x b_\alpha(\alpha'\beta x)a_x(aa^{\mathrm{V}}b)$$
$$-a'_{\alpha'}a''_{\beta}a^{\mathrm{V}}_\alpha b_x(\alpha'\beta x)a_x(aa^{\mathrm{V}}b)-a'_{\beta}a''_{\alpha'}a^{\mathrm{V}}_x b_\alpha(\alpha'\beta x)a_x(aa^{\mathrm{V}}b)$$
$$+a'_{\beta}a''_{\alpha'}a^{\mathrm{V}}_\alpha b_x(\alpha'\beta x)a_x(aa^{\mathrm{V}}b)$$
$$=0-(a'a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\overparen{\beta x})a'_{(a''a^{\mathrm{V}})}a''_{\beta}b_x a_x(aa^{\mathrm{V}}b)-0$$
$$+(a''a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\overparen{\beta x})a''_{(a^{\mathrm{V}}a')}a'_{\beta}b_x a_x(aa^{\mathrm{V}}b)$$
$$=-\tfrac{1}{3}D(\overparen{\beta x}a''a^{\mathrm{V}})a''_{\beta}b_x a_x(aa^{\mathrm{V}}b)+\tfrac{1}{3}D(\overparen{\beta x}a^{\mathrm{V}}a')a'_{\beta}b_x a_x(aa^{\mathrm{V}}b)$$
$$=-\tfrac{1}{3}D(a''_{\beta}a^{\mathrm{V}}_x-a''_x a^{\mathrm{V}}_{\beta})a''_{\beta}b_x a_x(aa^{\mathrm{V}}b)+\tfrac{1}{3}D(a^{\mathrm{V}}_{\beta}a'_x-a^{\mathrm{V}}_x a'_{\beta})a'_{\beta}b_x a_x(aa^{\mathrm{V}}b)$$
$$=\tfrac{1}{3}Da_x a''_x a''_{\beta}a^{\mathrm{V}}_{\beta}b_x(aa^{\mathrm{V}}b)+\tfrac{1}{3}Da^{\mathrm{V}}_{\beta}a'_x a'_{\beta}b_x a_x(aa^{\mathrm{V}}b)$$
$$=\tfrac{1}{6}Da''_x a''_{\beta}b_x(aa^{\mathrm{V}}b)\{a_x a^{\mathrm{V}}_{\beta}-a_{\beta}a^{\mathrm{V}}_x\}+\tfrac{1}{6}Da'_x a'_{\beta}b_x(aa^{\mathrm{V}}b)\{a^{\mathrm{V}}_{\beta}a_x-a^{\mathrm{V}}_x a_{\beta}\}$$
$$=\tfrac{1}{6}Da''_x a''_{\beta}b_x b_\alpha(\alpha x\beta)-\tfrac{1}{6}Da'_x a'_{\beta}b_x b_\alpha(\alpha\beta x)=-\tfrac{1}{6}DG-\tfrac{1}{6}DG=-\tfrac{1}{3}DG.$$

6. $(bab''')(\alpha\alpha'\beta)(\alpha\overparen{ab'''}x)b_x(\alpha'\beta x)$

$$=(a'_{\alpha'}a''_{\beta}-a'_{\beta}a''_{\alpha'})(\alpha'\beta x)(a_x b'''_\alpha-a_\alpha b'''_x)b_x(bab''')$$
$$=a'_{\alpha'}a''_{\beta}a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')-0-a'_{\beta}a''_{\alpha'}a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')+0$$
$$=2a'_{\alpha'}a''_{\beta}a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')=2(a'a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\overparen{\beta x})a'_{(a''b''')}a''_{\beta}a_x b_x(bab''')$$
$$=\tfrac{2}{3}D(\overparen{\beta x}a''b''')a''_{\beta}a_x b_x(bab''')=\tfrac{2}{3}D\{a''_{\beta}b'''_x-a''_x b'''_{\beta}\}a''_{\beta}a_x b_x(bab''')$$
$$=-\tfrac{2}{3}Da''_x b'''_{\beta}a''_{\beta}a_x b_x(bab''')=-\tfrac{2}{3}D(b'''b'b'')(b'b''a'')b'''_{(ba)}a''_x a_x b_x=0.$$

7. $a_{\beta'}(\overparen{a'''b}\alpha\beta)(\overparen{a'''b}\beta'x)a_x(\alpha\beta x)=a_{\beta'}a_x(a'''_\alpha b_\beta-a'''_\beta b_\alpha)(a'''_{\beta'}b_x-a'''_x b_{\beta'})(\alpha\beta x)$

$$=a_{\beta'}a_x(\alpha\beta x)[a'''_\alpha b_\beta a'''_{\beta'}b_x-a'''_\alpha b_\beta a'''_x b_{\beta'}-a'''_\beta b_\alpha a'''_{\beta'}b_x+a'''_\beta b_\alpha a'''_x b_{\beta'}]$$
$$=(a'''a'a'')(a'a''\overparen{\beta x})a'''_{\beta'}a'_{\beta}a_x b_\beta b_x-(a'''a'a'')(a'a''\overparen{\beta x})a'''_x b_\beta b_{\beta'}a_{\beta'}a_x$$
$$-a'''_\beta b_\alpha a'''_{\beta'}b_x a_{\beta'}a_x(\alpha\beta x)+(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a)b_\alpha a'''_x a'''_\beta a_x(\alpha\beta x)$$
$$=\tfrac{1}{3}D(\beta x\beta')a_{\beta'}a_x b_\beta b_x-0-A_{122}G+\tfrac{1}{3}D'(aa'a'')(a'a''\overparen{\beta x})a_x a'''_x a'''_\beta=-A_{122}G.$$

8. $b_{\beta'}(\overparen{ab'''}\alpha\beta)(\overparen{ab'''}\beta'x)b_x(\alpha\beta x)=(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\overline{x\overparen{ab'''}})b_x$, etc. $=0$.

9. $a_{\alpha'}(\overparen{a'''b}\alpha\beta)(\overparen{a'''b}\alpha'x)a_x(\alpha\beta x)=(aa'''a^{\mathrm{IV}})(a'''a^{\mathrm{I}}\overparen{x\beta})a_x(\beta\alpha\beta')(\alpha\beta'x)$

$$=\tfrac{1}{3}D(x\beta x),\ \text{etc.}\ =0.$$

10. $b_{\alpha'}(\overparen{ab'''}\alpha\beta)(\overparen{ab'''}\alpha'x)b_x(\alpha\beta x)=b_{\alpha'}b_x a_\beta b'''_\alpha a_x b'''_{\alpha'}(\alpha\beta x)$

But, $b_{\alpha'}b_x b'''_\alpha b'''_{\alpha'}a_\beta a_x(\alpha\beta x)-b'''^2_{\alpha'}b_x b_\alpha a_\beta a_x(\alpha\beta x)=b'''_{\alpha'}b_x a_\beta a_x(\alpha\beta x)[b_{\alpha'}b'''_\alpha-b_\alpha b'''_{\alpha'}]$

$$=b'''_{\alpha'}b_x a_\beta a_x(\alpha\beta x)(\overparen{bb'''}\alpha'\alpha)=\tfrac{1}{2}a_x b_x a_\beta(\overparen{bb'''}\alpha'\alpha)[(\alpha\beta x)\alpha'_{b'''}-(\alpha'\beta x)\alpha_{b'''}]$$
$$=\tfrac{1}{2}a_x b_x a_\beta(\overparen{bb'''}\alpha'\alpha)[(\alpha\beta\alpha')b'''_x-(\alpha x\alpha')b'''_\beta]=0.$$
$$\therefore\ b_{\alpha'}(\overparen{ab'''}\alpha\beta)(\overparen{ab'''}\alpha'x)b_x(\alpha\beta x)=A_{112}G.$$

11. $(aa^{\mathrm{IV}}b''')(\widehat{a'''b}\alpha\beta)(\widehat{a'''b}\widehat{a^{\mathrm{IV}}b'''}x)a_x(\alpha\beta x)=-\tfrac{1}{2}A_{112}G.$

For, $(aa^{\mathrm{IV}}b''')\{a'''_\alpha b_\beta-a'''_\beta b_\alpha\}(\alpha\beta x)a_x\{(a'''a^{\mathrm{IV}}b''')b_x-(ba^{\mathrm{IV}}b''')a'''_x\}$

$=(aa^{\mathrm{IV}}b''')\{a'''_\alpha b_\beta a_x(\alpha\beta x)-a'''_\beta b_\alpha a_x(\alpha\beta x)\}\{(a'''a^{\mathrm{IV}}b''')b_x-(ba^{\mathrm{IV}}b''')a'''_x\}.$

$(aa^{\mathrm{IV}}b''')a'''_\alpha b_\beta a_x(\alpha\beta x)(a'''a^{\mathrm{IV}}b''')b_x=0$

and $(aa^{\mathrm{IV}}b''')a'''_\alpha b_\beta a_x(\alpha\beta x)(ba^{\mathrm{IV}}b''')a'''_x=0.$

But, $-(aa^{\mathrm{IV}}b''')a'''_\beta b_\alpha a_x(\alpha\beta x)(a'''a^{\mathrm{IV}}b''')b_x$

$=-\tfrac{1}{2}b_\alpha a_x(\alpha\beta x)b_x(a'''a^{\mathrm{IV}}b''')[(aa^{\mathrm{IV}}b''')a'''_\beta-(aa'''b''')a^{\mathrm{IV}}_\beta]$

$=-\tfrac{1}{2}b_\alpha a_x(\alpha\beta x)b_x(a'''a^{\mathrm{IV}}b''')[(aa^{\mathrm{IV}}a''')b'''_\beta+(a^{\mathrm{IV}}b'''a''')a_\beta]$

$=\tfrac{1}{2}(aa'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}b''')a_xb_\alpha(\alpha\beta x)b_xb'''_\beta-\tfrac{1}{2}(a'''a^{\mathrm{IV}}ab''')^2b_\alpha a_xa_\beta b_x(\alpha\beta x)$

$=-\tfrac{1}{2}A_{112}G.$ Also $(aa^{\mathrm{IV}}b''')a'''_\beta b_\alpha a_x(\alpha\beta x)(ba^{\mathrm{IV}}b''')a'''_x$

$=\tfrac{1}{2}a'''_\beta a_xa'''_x(\alpha\beta x)(ba^{\mathrm{IV}}b''')\{(aa^{\mathrm{IV}}b''')b_\alpha-(aa^{\mathrm{IV}}b)b'''_\alpha\}$

$=\tfrac{1}{2}a'''_\beta a_xa'''_x(\alpha\beta x)(ba^{\mathrm{IV}}b''')\{(abb''')a^{\mathrm{IV}}_\alpha-(a^{\mathrm{IV}}b'''b)a_\alpha\}$

$=\tfrac{1}{2}a'''_\beta a_xa'''_xa^{\mathrm{IV}}_\alpha(\alpha\beta x)(ba^{\mathrm{IV}}b''')(abb''')+\tfrac{1}{2}(ba^{\mathrm{IV}}b''')^2a_\alpha a'''_\beta a_xa'''_x(\alpha\beta x)$

$=\tfrac{1}{2}(a^{\mathrm{V}}a'a'')(a'a''\widehat{\beta x})a^{\mathrm{IV}}_{(b'''b)}a'''_\beta a_xa'''_x(abb''')=\tfrac{1}{6}D(\widehat{\beta x}b'''b)a'''_\beta a_xa'''_x(abb''')$

$=\tfrac{1}{6}D\{b'''_\beta b_x-b'''_xb_\beta\}a'''_\beta a_xa'''_x(abb''')$

$=\tfrac{1}{6}D(b'''b'b'')(b'b''a''')b'''_{(ab)}b_xa_xa'''_x-\tfrac{1}{6}D(bb'b'')(b'b''a''')b_{(b'''a)}b''_xa_xa'''_x$

$=\tfrac{1}{18}DD'(a'''ab)a_xa'''_xb_x-\tfrac{1}{18}DD'(a'''b'''a)a'''_xa_xb'''_x=0.$

12. $(ba'''b^{\mathrm{IV}})(\widehat{ab'''}\alpha\beta)(\widehat{ab'''}\widehat{a'''b^{\mathrm{IV}}}x)b_x(\alpha\beta x)$

$=(ba'''b^{\mathrm{IV}})\{a_\alpha b'''_\beta-a_\beta b'''_\alpha\}\beta_{(x\alpha)}\{(aa'''b^{\mathrm{IV}})b'''_x-(b'''a'''b^{\mathrm{IV}})a_x\}b_x$

$=0-0-(ba'''b^{\mathrm{IV}})a_\beta b'''_\alpha(\alpha\beta x)(aa'''b^{\mathrm{IV}})b'''_xb_x$

$+(ba'''b^{\mathrm{IV}})a_\beta b'''_\alpha(\alpha\beta x)(b'''a'''b^{\mathrm{IV}})a_xb_x$

$=-\tfrac{1}{2}b_xb'''_xb'''_\alpha(\alpha\beta x)(aa'''b^{\mathrm{IV}})\{(ba'''b^{\mathrm{IV}})a_\beta-(bab^{\mathrm{IV}})a'''_\beta\}$

$+\tfrac{1}{2}a_\beta(\alpha\beta x)(b'''a'''b^{\mathrm{IV}})a_xb_x\{(ba'''b^{\mathrm{IV}})b'''_\alpha-(ba'''b''')b^{\mathrm{IV}}_\alpha\}$

$=-\tfrac{1}{2}b_xb'''_xb'''_\alpha(\alpha\beta x)(aa'''b^{\mathrm{IV}})\{(ba'''a)b^{\mathrm{IV}}_\beta+(a'''b^{\mathrm{IV}}a)b_\beta\}$

$+\tfrac{1}{2}a_\beta(\alpha\beta x)(b'''a'''b^{\mathrm{IV}})a_xb_x\{(bb'''b^{\mathrm{IV}})a'''_\alpha-(a'''b'''b^{\mathrm{IV}})b_\alpha\}$

$=0+0+0+\tfrac{1}{2}(a'''b'''b^{\mathrm{IV}})^2a_\beta a_xb_xb_\alpha(\alpha\beta x)=\tfrac{1}{2}A_{122}G.$

13. $a_{\beta''}(\beta\alpha\beta')(\beta\beta''x)a_x(\alpha\beta'x)=a_{\beta''}\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}\{b'_{\beta''}b''_x-b'_xb''_{\beta''}\}(\beta'x\alpha)a_x$

$=0-a_{\beta''}a_xb'_\alpha b'_x(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b''_{\beta''}$

$-a_{\beta''}a_x(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b'_{\beta''}b''_\alpha b''_x+0$

$=-\tfrac{1}{3}a_{\beta''}a_xb'_\alpha b'_xD'(x\alpha\beta'')-\tfrac{1}{3}a_{\beta''}a_xb''_\alpha b''_xD'(x\alpha\beta'')=-\tfrac{2}{3}D'G.$

14. $b_{\beta''}(\beta\alpha\beta')(\beta\beta''x)b_x(\alpha\beta'x)=(bb^{\mathrm{V}}b^{\mathrm{VI}})(b^{\mathrm{V}}b^{\mathrm{VI}}\widehat{x\beta})b_x(\beta\alpha\beta')(\alpha\beta'x)=0.$

15. $a_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)a_x(\alpha\beta'x)=(aa'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\widehat{x\beta})a_x(\beta\alpha\beta')(\alpha\beta'x)=0.$

16. $$\begin{aligned}&b_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)b_x(\alpha\beta'x)=b_{\alpha'}\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}\{b'_{\alpha'}b''_x-b'_xb''_{\alpha'}\}(\beta'x\alpha)b_x\\
&=0-(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b''_{\alpha'}b'_\alpha b'_xb_xb_{\alpha'}\\
&\quad-(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b'_{\alpha'}b''_\alpha b''_xb_xb_{\alpha'}+0=-\tfrac{1}{3}D'(x\alpha\alpha')b'_\alpha b'_xb_xb_{\alpha'}\\
&\quad-\tfrac{1}{3}D'(x\alpha\alpha')b''_\alpha b''_xb_xb_{\alpha'}=-\tfrac{2}{3}D'(\alpha\alpha'x)b'_\alpha b_{\alpha'}b'_xb_x\\
&=-\tfrac{2}{3}D'(a'_{\alpha'}a''_x-a'_xa''_{\alpha'})b'_\alpha b_{\alpha'}b_xb'_x=-\tfrac{2}{3}D'(a'a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}b)a'_{(a''b')}a''_xb_xb'_x\\
&\quad+\tfrac{2}{3}D'(a''a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}b)a''_{(b'a')}b_xa'_xb'_x=-\tfrac{2}{9}DD'(ba''b')b_xb'_xa''_x\\
&\quad+\tfrac{2}{9}DD'(bb'a')b_xb'_xa'_x=0.\end{aligned}$$

17. $$\begin{aligned}&(aa'''b)(\beta\alpha\beta')(\beta\widehat{a'''b}x)a_x(\alpha\beta'x)\\
&=(aa'''b)(\alpha\beta'x)a_x\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}\{a'''_xb_\beta-a'''_\beta b_x\}\\
&=(aa'''b)(\alpha\beta'x)a_x\{b'_\alpha b''_{\beta'}a'''_xb_\beta-b'_\alpha b''_{\beta'}a'''_\beta b_x-b'_{\beta'}b''_\alpha a'''_xb_\beta+b'_{\beta'}b''_\alpha a'''_\beta b_x\}\\
&=(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b''_{(bb')}a'''_xb'_\alpha a_x(aa'''b)\\
&\quad-(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b''_{(a'''b')}b_xb'_\alpha a_x(aa'''b)\\
&\quad-(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b'_{(b''b)}a'''_xb''_\alpha a_x(aa'''b)\\
&\quad+(b'b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b'_{(b''a''')}b_xb''_\alpha a_x(aa'''b)\\
&=\tfrac{1}{3}D'(bb'\widehat{x\alpha})b'_\alpha a_xa'''_x(aa'''b)-\tfrac{1}{3}D'(a'''b'\widehat{x\alpha})b_xb'_\alpha a_x(aa'''b)\\
&\quad-\tfrac{1}{3}D'(b''b\widehat{x\alpha})b''_\alpha a_xa'''_x(aa'''b)+\tfrac{1}{3}D'(b''a'''\widehat{x\alpha})b_xb''_\alpha a_x(aa'''b)\\
&=-\tfrac{2}{3}D'\{a'''_xb'_\alpha-a'''_\alpha b'_x\}b_xb'_\alpha a_x(aa'''b)=-\tfrac{2}{3}D'b'^2_\alpha a'''_xb_xa_x(aa'''b)\\
&\quad+\tfrac{2}{3}D'a'''_\alpha b'_xb_xb'_\alpha a_x(aa'''b)=\tfrac{2}{3}D'(a'''a'a'')(a'a''b')a'''_{(ba)}b'_xb_xa_x\\
&=\tfrac{2}{9}DD'(b'ba)b'_xb_xa_x=0.\end{aligned}$$

18. $$\begin{aligned}&(bab^{\mathrm{V}})(\beta\alpha\beta')(\widehat{\beta ab^{\mathrm{V}}}x)b_x(\alpha\beta'x)\\
&=(bab^{\mathrm{V}})(\widehat{\beta ab^{\mathrm{V}}}x)b_x\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}\{a'_{\beta'}a''_x-a'_xa''_{\beta'}\}\\
&=(bab^{\mathrm{V}})b_x[b'_\alpha b''_{\beta'}a'_{\beta'}a''_x-b'_\alpha b''_{\beta'}a'_xa''_{\beta'}-b'_{\beta'}b''_\alpha a'_{\beta'}a''_x+b'_{\beta'}b''_\alpha a'_xa''_{\beta'}](a_xb^{\mathrm{V}}_\beta-a_\beta b^{\mathrm{V}}_x)\\
&=(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a')b''_{(b^{\mathrm{V}}b')}b'_\alpha a''_xa_x(bab^{\mathrm{V}})b_x\\
&\quad-(b''b'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a')b''_{(ab')}b'_\alpha a''_xb^{\mathrm{V}}_x(bab^{\mathrm{V}})b_x,\ \text{etc.}\\
&=\tfrac{1}{3}D'(a'b^{\mathrm{V}}b')b'_\alpha a''_xa_xb_x(bab^{\mathrm{V}})-\tfrac{1}{3}D'(a'ab')b'_\alpha a''_xb^{\mathrm{V}}_xb_x(bab^{\mathrm{V}}),\ \text{etc.}\\
&=\tfrac{1}{6}D'a''_xa_xb_x(a'b^{\mathrm{V}}b')[(bab^{\mathrm{V}})b'_\alpha-(bab')b^{\mathrm{V}}_\alpha]-0,\ \text{etc.}\\
&=\tfrac{1}{6}D'a''_xa_xb_x(a'b^{\mathrm{V}}b')[(ab^{\mathrm{V}}b')b_\alpha-(bb^{\mathrm{V}}b')a_\alpha],\ \text{etc.}\\
&=\tfrac{1}{6}D'a''_xa_xb_x(a'b^{\mathrm{V}}b')(ab^{\mathrm{V}}b')b_\alpha,\ \text{etc.}=\tfrac{1}{6}D'a''_xa_xb_xa'_\beta a_\beta b_\alpha,\ \text{etc.}\\
&=\tfrac{1}{12}D'a_xb_xa_\beta b_\alpha[a''_xa'_\beta-a'_xa''_\beta],\ \text{etc.}=\tfrac{1}{12}D'a_xb_xa_\beta b_\alpha(\alpha\beta x),\ \text{etc.}\\
&=\tfrac{1}{12}D'G+\tfrac{1}{12}D'G+\tfrac{1}{12}D'G+\tfrac{1}{12}D'G=\tfrac{1}{3}D'G.\end{aligned}$$

Combining these results as in (3) we obtain G'.

TABLE II.

System of $a_x^2+kb_x^2,\ (\alpha\beta x)^2$.

1. $K \quad = a_x^2+kb_x^2.$

2. $F \quad = (\alpha\beta x)^2.$

3. $(L)' = \alpha_u^2+2k(abu)^2+k^2\beta_u^2.$

4. $(L')' = \frac{4}{9}\{3DA_{122}u_\beta^2+3D'A_{112}u_\alpha^2-2DD'(abu)^2\}.$

5. $(D)' = D+3kA_{112}+3k^2A_{122}+k^3D'.$

6. $(D')' = \frac{8}{27}\{9A_{112}A_{122}-DD'\}DD'.$

7. $\Phi' \quad = \frac{1}{3}DL'+A_{122}L+k\{\frac{1}{3}D'L+A_{112}L'\}.$

8. $C_1' \quad = C_8-kC_5.$

9. $A_{122}' = \frac{4}{9}\{3DA_{122}^2+DD'A_{112}+k(3D'A_{112}^2+DD'A_{122})\}.$

10. $C_2' \quad = 4\{3DA_{122}C_2+DD'C_3+k(3D'A_{112}C_3+DD'C_2)\}.$

11. $C_3' \quad = (\frac{2}{3}DA_{122}u_x-\frac{2}{3}DC_2)+2k(\frac{1}{9}DD'u_x-A_{112}C_2+A_{112}A_{122}u_x-A_{122}C_8)$
$+k^2(\frac{2}{3}D'A_{112}u_x-\frac{2}{3}D'C_8).$

12. $A_{112}' = \frac{4}{3}DA_{122}+2k(\frac{1}{3}DD'+A_{112}A_{122})+k^2\cdot\frac{4}{3}D'A_{112}.$

13. $C_4' \quad = \frac{4}{9}[3DA_{122}C_4-2DD'C_7+6k(DA_{122}C_6-D'A_{112}C_7)$
$-k^2(3D'A_{112}C_4-2DD'C_6)].$

14. $F' \quad = \frac{4}{9}\{3DA_{122}F+2DD'A_{112}f-\frac{2}{3}D^2D'g$
$+2k(3DA_{122}^2g-DD'A_{122}f-A_{112}DD'g+3A_{112}^2D'f+DD'F)$
$+k^2(3D'A_{112}F+2DD'A_{122}g-\frac{2}{3}DD'^2f)\}.$

15. $C_5' \quad = \frac{4}{9}\{2DD'A_{112}C_8+\frac{2}{3}D^2D'C_5$
$+2k(3D'A_{112}^2C_8+DD'A_{112}C_5-3DA_{122}^2C_5-DD'A_{122}C_8)$
$+k^2(-2DD'A_{122}C_5-\frac{2}{3}DD'^2C_8)\}.$

16. $C_6' \quad = \frac{4}{9}\{3DA_{122}^2C_4-2DD'A_{122}C_7-DD'A_{112}C_4+\frac{2}{3}D^2D'C_6$
$+k(-3D'A_{112}^2C_4+2DD'A_{112}C_6+DD'A_{122}C_4-\frac{2}{3}DD'^2C_7)\}.$

17. $C_7' \quad = \frac{1}{3}DC_4+k(A_{112}C_4-2A_{122}C_7+\frac{2}{3}DC_6)$
$+k^2(-A_{122}C_4+A_{112}C_6-\frac{2}{3}D'C_7)+k^3(-\frac{1}{3}D'C_4).$

18. $C_8' \quad = \frac{4}{9}\{3DA_{122}C_8-\frac{2}{3}D^2D'C_1+k(6DA_{122}^2C_1-4DD'A_{112}C_1+2DD'C_8-3DA_{122}C_5)$
$+k^2(-6D'A_{112}^2C_1+4DD'A_{122}C_1-2DD'C_5+3D'A_{112}C_8)$
$+k^3(\frac{2}{3}DD'^2C_1-3D'A_{112}C_5)\}.$

19. $G' \quad = \frac{4}{9}\{\frac{2}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)$
$+k^2(6D'A_{112}^2-4DD'A_{122})-\frac{2}{3}DD'^2\cdot k^3\}G.$

20. $\Gamma' \quad = \frac{4}{9}\{\frac{2}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)$
$+k^2(6D'A_{112}^2-4DD'A_{122})-\frac{2}{3}DD'^2\cdot k^3\}\Gamma.$

§ 3. SYSTEM OF $a^2_x+kb^2_x$, c^2_x.

TABLE III.

$(a^2_x+kb^2_x,\, c^2_x)^{ij}_{kl}$ *in terms of the system of Baker.*

$\Phi'' = (a^2_x+kb^2_x,\, c^2_x)^{02}_{00} = \Phi_2+k\Phi_1.$

$C''_1 = (a^2_x+kb^2_x,\, c^2_x)^{01}_{00} = C_{12}+kC_{11}.$

$A''_{122} = (a^2_x+kb^2_x,\, \gamma^2_u)^{20}_{00} = A_1+kA_2.$

$C''_2 = (a^2_x+kb^2_x,\, \gamma^2_u)^{10}_{00} = C_{21}+kC_{22}.$

$C''_3 = [\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u,\, c^2_x]^{00}_{10} = C_{23}+2kM+k^2C_{24}.$

$A''_{112} = [\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u,\, c^2_x]^{00}_{20} = A_3+2kN+k^2A_4.$

$C''_4 = [\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u,\, \gamma^2_u]^{00}_{01} = C_{42}+2k(O_1-O_2)+k^2C_{41}.$

$F'' = [\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u,\, \gamma^2_u]^{00}_{02} = F_2+2k(A_1g-2P+A_2f)+k^2F_1.$

$C''_5 = [\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u,\, c^2_x\gamma^2_u]^{00}_{11} = C_{51}+2k(Q_1-Q_2)+k^2C_{52}.$

$C''_6 = (a^2_x+kb^2_x,\, c^2_x\gamma^2_u)^{11}_{00} = C_{61}+kC_{62}.$

$C''_7 = [(a^2_x+kb^2_x)(\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u),\, c^2_x]^{01}_{10}$
$= C_{63}+k(2R_1-S_1J+2O_3)+k^2(2R_2-S_2J+2O_4)+k^3C_{64}.$

$C''_8 = [(a^2_x+kb^2_x)(\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u),\, \gamma^2_u]^{10}_{01}$
$= C_{53}+k(2A_1C_1-2fS_3+3T_1-T_2-JU_1)$
$+k^2(-2A_2C_1+2gS_3+3T_3-T_4-JU_2)+k^3C_{54}.$

$G'' = [(a^2_x+kb^2_x)(\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u),\, c^2_x\gamma^2_u]^{10}_{11}$
$= G_1+k[2A_1V-2fW+X_1]+k^2[-2A_2V+2gW+X_2]+k^3G_2.$

$\Gamma'' = [(a^2_x+kb^2_x)(\alpha^2_u+2k(ab)^2_u+k^2\beta^2_u),\, c^2_x\gamma^2_u]^{11}_{10}$
$= \Gamma_2+k[2A_1Y+S_1L''-2S_3\Phi_2+Z_1]+k^2[-2A_2Y-S_2L''+2S_3\Phi_1+Z_2]+k^3\Gamma_1.$